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Clebsch–Gordan coefficients of SU(3) with simple symmetry properties

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Abstract. Using an appropriate labelling operator constructed from representation generators, SU(3) Clebsch-Gordan coefficients are introduced whose symmetry properties are similar to those of their SU(2) counterparts. An algebraic algorithm for computing the coefficients is presented.

1. Introduction

The SU(3) Racah-Wigner algebra, frequently used in modern theoretical physics, has been the subject of numerous investigations in the last two decades (cf, e.g. Edmonds 1962, Moshinsky 1962, 1963, de Swart 1963, Hecht 1965, Chew and Sharp 1967, Resnikoff 1967a, b, Draayer and Akiyama 1973, Millener 1978, Klimyk 1979). Its large-scale applications, mainly in nuclear theory, employ the SU(3) Clebsch-Gordan coefficients introduced by Draayer and Akiyama (1973). These authors use the theory of Wigner operators, developed for general unitary groups by Biedenharn, Louck and co-workers (Chacón *et al* 1972 and references therein). Their method of constructing the SU(3) Wigner operators, however, is rather asymmetric in the representations involved, and thus their coefficients have rather complex symmetry properties. The potential advantages of coefficients of more simple symmetry properties, similar to those known from the SU(2) case, are obvious. The existence of such coefficients, by no means trivial, has been discussed by Derome (1967) in the context of his general analyses of the Racah-Wigner algebras (Derome and Sharp 1965, Derome 1966), and briefly re-discussed in the same context by Butler (1975).

In this paper, we present an explicit construction of SU(3) Clebsch-Gordan coefficients with simple symmetry properties, based on applying an appropriate labelling operator constructed from representation generators. The construction is purely infinitesimal and makes use of the SU(3) projection technique.

2. Conventions and notation

Only the ket representations will be considered. The irreducible representation of highest weight (ab) (IR of HW (ab)) will be denoted by $D^{(ab)}$; its basic generators will be denoted by $E_{\rho\sigma}$, $\rho, \sigma = 1, 2, 3$, and defined by the usual defining relations:

$$[E_{\rho\sigma}, E_{\rho'\sigma'}] = \delta_{\rho'\sigma}E_{\rho\sigma'} - \delta_{\rho\sigma'}E_{\rho'\sigma} \tag{2.1}$$

$E_{11}+E_{22}+E_{33}=0$ and $E^{\dagger}_{\rho\sigma}=E_{\sigma\rho}$. As the weight operators we take

$$W_1(E)=E_{11}-E_{22} \qquad W_2(E)=E_{22}-E_{33}. \tag{2.2}$$

Other operators to be used are the Casimir operators:

$$F_2(E)=\tfrac{3}{2}\,\mathrm{Tr}(EE) \qquad F_3(E)=9\,\mathrm{Tr}(EEE)-\tfrac{27}{2}\,\mathrm{Tr}(EE) \tag{2.3}$$

where $\mathrm{Tr}(EE)=\Sigma\, E_{\rho\sigma}E_{\sigma\rho}$; their respective eigenvalues in the states of $D^{(ab)}$ are (cf, e.g. de Swart 1963)

$$f_2(ab)=(a+b+3)(a+b)-ab \qquad f_3(ab)=(a-b)(2a+b+3)(a+2b+3). \tag{2.4}$$

In addition to the above operators we shall use the 'isospin' operators:

$$I_+(E)=E_{12} \qquad I_z(E)=\tfrac{1}{2}(E_{11}-E_{22}) \qquad I_-(E)=E_{21} \tag{2.5}$$

referring to the SU(2) isospin subgroup, the 'angular momentum' operators

$$\Lambda_+(E)=\sqrt{2}(E_{12}+E_{23}) \qquad \Lambda_z(E)=E_{11}-E_{33} \qquad \Lambda_-(E)=\sqrt{2}(E_{21}+E_{32}) \tag{2.6}$$

referring to the SO(3) angular momentum subgroup, the 'hypercharge' operator

$$Y(E)=\tfrac{1}{3}(E_{11}+E_{22}-2E_{33}) \tag{2.7}$$

and the irreducible tensorial operators of the SU(2) isospin group $B^{(k)}_{k_z}(E)$ defined by

$$B^{(k)}_{k_z}(E)=(E_{32})^{k+k_z}(-E_{31})^{k-k_z}\binom{2k}{k+k_z}^{1/2}. \tag{2.8}$$

The highest state of $D^{(ab)}$ will be denoted by $|ab\rangle$, and defined by $W_1(E)|ab\rangle=a|ab\rangle$, $W_2(E)|ab\rangle=b|ab\rangle$ and $E_{\rho\sigma}|ab\rangle=0$ for $\rho<\sigma$; its hypercharge, isospin and angular momentum will be denoted respectively by y_0, i_0 and λ_0; explicitly,

$$y_0=\tfrac{1}{3}(a+2b) \qquad i_0=\tfrac{1}{2}a \qquad \lambda_0=a+b. \tag{2.9}$$

The canonical basic states of $D^{(ab)}$ of hypercharge y, isospin i and isospin projection i_z will be denoted by $|ab\,yii_z\rangle$. Their relative phases are assumed to be in agreement with the phase convention of Baird and Biedenharn (1963). The canonical states satisfy the recursion relation ($y<y'$):

$$|ab\,yii_z\rangle=\sum_{i'k_zi'_z}B^{(k)}_{k_z}(E)|ab\,y'i'i'_z\rangle(kk_z\,i'i'_z|ii_z)(-1)^{i_0+\varkappa+i} \times[(2\varkappa+1)(2i'+1)]^{1/2}(N^{(ab)}_{yi}/N^{(ab)}_{y'i'})\begin{Bmatrix} i_0 & \varkappa & i \\ k & i' & \varkappa' \end{Bmatrix} \tag{2.10}$$

where $k=\tfrac{1}{2}(y'-y)$, $N^{(ab)}_{yi}$ is the normalisation factor given by

$$N^{(ab)}_{yi}=\{[(i_0+b-\varkappa+i+1)!(i_0+b-\varkappa-i)!]/[(a+b+1)!b!(2\varkappa)!]\}^{1/2} \tag{2.11}$$

$\varkappa=\tfrac{1}{2}(y_0-y)$, and the round-bracket symbol is the SU(2) Clebsch–Gordan coefficient. The reduced matrix elements of the tensors $B^{(k)}_{k_z}(E)$ between the canonical states are (again $\varkappa=\tfrac{1}{2}(y_0-y)$)

$$\langle ab\,yi\|B^{(k)}\|ab\,y'i'\rangle =\delta_{\varkappa,k+\varkappa'}(-1)^{i_0+k+\varkappa+i'}[(2\varkappa+1)(2i+1)(2i'+1)]^{1/2}(N^{(ab)}_{y'i'}/N^{(ab)}_{yi}) \times\begin{Bmatrix} i_0 & \varkappa & i \\ k & i' & \varkappa' \end{Bmatrix}. \tag{2.12}$$

The dimension of $D^{(ab)}$ will be denoted by $\dim(ab)$.

The conjugate IR generated by the operators $\bar{E}_{\rho\sigma}=-E_{\sigma\rho}$ will be denoted by $\bar{D}^{(ba)}$; the canonical basic states of $D^{(ab)}$ and $\bar{D}^{(ba)}$ are related by (cf, e.g. de Swart 1963)

$$|ab\,yii_z\rangle=\overline{|ba\,-yi-i_z\rangle}(-1)^{a+b+\frac{3}{2}y+i_z}. \tag{2.13}$$

3. The labelling operator and the *s*-classified reduced states

We solve the multiplicity problem arising in reducing the product $D^{(a'b')}\times D^{(a''b'')}$ by introducing the *s*-classified reduced states satisfying the eigenproblem

$$S(E', E'')|(a'b'\ a''b'')ab\, yii_z\, s\rangle = s|(a'b'\ a''b'')ab\, yii_z\, s\rangle \tag{3.1}$$

with $S(E', E'')$ being the labelling operator introduced in the following way. We require this operator to be Hermitean, invariant under the SU(3) transformations generated by the operators $E_{\rho\sigma}=E'_{\rho\sigma}+E''_{\rho\sigma}$, and of the lowest possible order in the generators $E'_{\rho\sigma}$ and $E''_{\rho\sigma}$. In addition we require this operator to be of the form ensuring the envisaged simple symmetry properties of the *s*-classified SU(3) Clebsch-Gordan coefficients. This requirement will be formulated as a symmetry relation imposed on the labelling operators to be used in reducing the triple product $D^{(a'b')}\times D^{(a''b'')}\times D^{(a'''b''')}$ and its conjugate. Explicitly this will be done by requiring that

$$S(E', E'') = -S(E'', E') = -S(E', E''') = -S(\bar{E}', \bar{E}'') \tag{3.2}$$

when applied to the SU(3) invariants of the triple products under consideration. The above requirements determine the labelling operator uniquely up to a factor. In an appropriate normalisation

$$S(E', E'') = 27\,\mathrm{Tr}(E'E''E'') - 27\,\mathrm{Tr}(E''E'E') - 2F_3(E') + 2F_3(E'') \tag{3.3}$$

where $F_3(E')$ is the third order Casimir operator of $D^{(a'b')}$ defined by (2.3). It should be noted that the same requirements, but with the positive sign everywhere in (3.2), cannot be met by an operator of the third order in the representation generators. The eigenproblem (3.1), as shown below, defines the *s*-classified reduced states unambiguously to within a phase.

The non-trivial part of determining the *s*-classified reduced states consists in constructing the highest ones, from which the others may easily be obtained by the lowering procedure (2.10). The highest states $|(a'b'\ a''b'')ab\, s\rangle$ are to be found by solving the eigenproblem (3.1) in the invariant subspace of $S(E', E'')$ formed by the highest states of $D^{(a'b')}\times D^{(a''b'')}$ of weight (ab). It can be shown that a suitable basis of this space is provided by the ξ states $|\xi_m\rangle$, $p\leqslant m\leqslant q$, defined by

$$|\xi_m\rangle = P^{(ab)}(E)\frac{(E''_{32})^{l+m}(-E''_{31})^{l-m}(E''_{21})^{n+m}}{(l+m)!(l-m)!(n+m)!}|a'b'\ a''b''\rangle \tag{3.4}$$

where $P^{(ab)}(E)$ is the subspace projector, $|a'b'\ a''b''\rangle = |a'b'\rangle|a''b''\rangle$, and l and n are the quantum numbers specified by

$$l=\tfrac{1}{6}(a'+a''-a+2b'+2b''-2b) \qquad n=\tfrac{1}{2}(a'+a''-a). \tag{3.5}$$

The limits of m are explicitly determined by

$$p=-\min(l, n) \qquad q=p+\mu-1 \tag{3.6}$$

with μ being the multiplicity of the irreducible components of the HW (ab) in the product. An explicit expression for the subspace projector $P^{(ab)}(E)$ to be used in (3.4) is given by (Smirnov 1969, Asherova *et al* 1971)

$$\begin{aligned} P^{(ab)}(E) &= P^{[a/2]}(E_{21})P^{[(a+b+1)/2]}(E_{31})P^{[b/2]}(E_{32}) \\ P^{[j]}(E_{\rho\sigma}) &= \sum_t (E_{\rho\sigma})^t(E^{\dagger}_{\rho\sigma})^t\frac{(-1)^t(2j+1)!}{t!(2j+1+t)!}. \end{aligned} \tag{3.7}$$

Employing the commutativity of $S(E', E'')$ and $P^{(ab)}(E)$ and the fact that $P^{(ab)}(E)E_{\rho\sigma}=0$ for $\rho>\sigma$, one finds that in the biorthonormal basis formed by the states $|\xi_m\rangle$ and by the dual states $|\xi_m^d\rangle$ the matrix of $S(E', E'')$ takes the quasi-tridiagonal form defined by $(S_{\bar{m}n}=\langle\xi_{\bar{m}}|S(E', E'')|\xi_m^d\rangle)$

$$(S)=\begin{pmatrix} \alpha_p & \beta_{p+1} & 0 & \cdots & 0 & 0 & 0 \\ \gamma_p & \alpha_{p+1} & \beta_{p+2} & \cdots & 0 & 0 & 0 \\ 0 & \gamma_{p+1} & \alpha_{p+2} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \alpha_{q-2} & \beta_{q-1} & 0 \\ 0 & 0 & 0 & \cdots & \gamma_{q-2} & \alpha_{q-1} & \beta_q \\ \omega_p & \omega_{p+1} & \omega_{p+2} & \cdots & \omega_{q-2} & \omega_{q-1} & \omega_q \end{pmatrix} \tag{3.8}$$

the values of the matrix elements being

$$\begin{aligned}\alpha_m = 18\{[(a+b-a'-b'+l-m+1)(a'-b'+3l+3m+6)\\ +(a'+2b'+6)(l+m+1)](l-m)\\ -(b-b'+l+m+1)(2a'+b'-3)(l+m)\\ +(a-a'+n-m+1)(a'+2b'-3l-3m+6)(n+m)\}+f_3(ab)-3f_3(a'b')\\ +f_3(a''b'')+27[f_2(ab)-f_2(a'b')-f_2(a''b'')]\\ -6\{(2a+b-2a'-b')[(a')^2-(b')^2+6a'+3b']\\ +b'(a-b-a'+b')(2a'+b'-3)\}\end{aligned}$$

$$\beta_m = 54(a+b+l-m+3)(l+m)(n+m)$$

$$\gamma_m = 54(a-a'+n-m)(b'-l-m)(l-m)$$

$$\omega_m = \delta_{mq}\alpha_q + \delta_{m,q-1}\gamma_{q-1} - \beta_{q+1}\begin{pmatrix} r-m \\ r-q-1\end{pmatrix}.$$

In the expression for ω_m

$$r=\min(l, b'-l, a''-n). \tag{3.9}$$

It should be noted that $r\leqslant q=\min(r, r+a'-l-n, r+b''-2l)$.

Since the matrix elements β_m are different from zero, the subspace is cyclic and, consequently, the eigenvalues s of (3.8) are all distinct. Thus the eigenproblem (3.1) determines the s-classified reduced states uniquely up to a phase. The remaining phase ambiguity will be removed by imposing the phase condition

$$\langle a'b'\ a''b''\ \lambda_0''\lambda_{0z}''|(a'b'\ a''b'')ab\,s\rangle>0 \tag{3.10}$$

requiring the s-classified highest state $|(a'b'\ a''b'')ab\,s\rangle$ to have a positive projection along the state $|a'b'\ a''b''\ \lambda_0''\lambda_{0z}''\rangle=|a'b'\rangle|a''b''\ \lambda_0''\lambda_{0z}''\rangle$, where $|a''b''\ \lambda_0''\lambda_{0z}''\rangle$ is the state of $D^{(a''b'')}$ of angular momentum $\lambda_0''=a''+b''$, of angular momentum projection $\lambda_{0z}''=a+b-a'-b'$ and of the usual relative phase with respect to the highest state $|a''b''\rangle=|a''b''\ \lambda_0''\lambda_0''\rangle$. The labels s are, in general, neither integral nor rational.

4. The s-classified SU(3) Clebsch–Gordan coefficients: the computational algorithm

By definition, the s-classified SU(3) Clebsch–Gordan coefficients $(a'b'\ y'i'i_z'\ a''b''\ y''i''i_z''|ab\ yii_z\ s)$ are the transformation coefficients between the s-classified SU(3) reduced states $|(a'b'\ a''b'')ab\ yii_z\ s\rangle$ and the unreduced states

$|a'b'\ y'i'i'_z\rangle|a''b''\ y''i''i''_z\rangle$. The non-trivial part of computing the coefficients consists in evaluating their isoscalar factors (cf Edmonds 1962) $(a'b'\ y'i'\ a''b''\ y''i''\|ab\ yi\ s)$, identical with the transformation coefficients between the s-classified SU(3) reduced states $|(a'b'\ a''b'')ab\ yii\ s\rangle$ and the SU(2) coupled states $|(a'b'\ y'i'\ a''b''\ y''i'')ii\rangle$. All s-classified SU(3) Clebsch-Gordan coefficients (CGCs) and their isoscalar factors (IFs) are real.

Typically all s-classified IFs of given $(a'b')$, $(a''b'')$ and (ab) are to be evaluated. The computational algorithm may then be as follows. First the allowed s and the coefficients $\langle\xi_m|(a'b'\ a''b'')ab\ s\rangle$ are to be found by solving the μ-dimensional matrix eigenproblem (see (3.1) and (3.8))

$$\sum_m (S_{\tilde{m}m} - s\delta_{\tilde{m}m})\langle\xi_m|(a'b'\ a''b'')ab\ s\rangle = 0 \tag{4.1}$$

with the subsidiary condition (cf (3.10))

$$\sum_m \langle a'b'\ a''b''\ \lambda''_0\lambda''_{0z}|\xi^d_m\rangle\langle\xi_m|(a'b'\ a''b'')ab\ s\rangle > 0 \tag{4.2}$$

the explicit expression for the first factor in (4.2), derivable by the conventional expansion technique, being

$$\langle a'b'\ a''b''\ \lambda''_0\lambda''_{0z}|\xi^d_m\rangle = \left(2^{m-l} + \sum_{\tilde{m}} C_{m\tilde{m}} 2^{\tilde{m}-l}\right)\left[2^{\lambda''_0-\lambda''_0} \Big/ \binom{2\lambda''_0}{\lambda''_0+\lambda''_{0z}}\right]^{1/2}$$

where

$$C_{m\tilde{m}} = (-1)^{\tilde{m}-q}\binom{r-m}{r-\tilde{m}}\binom{\tilde{m}-m-1}{q-m}. \tag{4.3}$$

The correct normalisation of the coefficients is easily determined by fitting to the normalisation conditions imposed on the IFs.

As the next step the 'highest' IFs of $y = y_0$ and $i = i_0$ are to be computed, making use of the relation $(\tilde{y}'' = y_0 - y'_0,\ \tilde{k}'' = \frac{1}{2}(y'' - \tilde{y}''))$

$$\begin{aligned}(a'b'\ y'i'\ a''b''\ y''i''\|ab\ y_0i_0\ s) &= N^{(a'b')}_{y'i'}(2i'+1)^{1/2}\sum_{\tilde{i}''}\langle a''b''\ \tilde{y}''\tilde{i}''\|B^{(\tilde{k}'')}\|a''b''\ y''i''\rangle\begin{Bmatrix} i_0 & i'_0 & \tilde{i}'' \\ \tilde{k}'' & i'' & i'\end{Bmatrix}\\ &\quad\times(-1)^{i'+\tilde{i}''+i_0+\tilde{k}''}(a'b'\ y'_0i'_0\ a''b''\ \tilde{y}''\tilde{i}''\|ab\ y_0i_0\ s)\end{aligned} \tag{4.4}$$

expressing the general highest IFs in terms of those with $y' = y'_0$ and $i' = i'_0$, and of the formula

$$\begin{aligned}(a'b'\ y'_0i'_0\ a''b''\ y''i''\|ab\ y_0i_0\ s) &= \sum_m \langle(a'b'\ y'_0i'_0\ a''b''\ y''i'')i_0i_0|\xi^d_m\rangle\langle\xi_m|(a'b'\ a''b'')ab\ s\rangle\end{aligned} \tag{4.5}$$

expressing these special highest IFs in terms of the coefficients $\langle\xi_m|(a'b'\ a''b'')ab\ s\rangle$; the first factor in (4.5) is

$$\langle(a'b'\ y'_0i'_0\ a''b''\ y''i'')i_0i_0|\xi^d_m\rangle = \left(\zeta_m + \sum_{\tilde{m}} C_{m\tilde{m}}\zeta_{\tilde{m}}\right)(i'_0i'_0i''\ i_0-i'_0|i_0i_0)^{-1}$$

with $C_{m\tilde{m}}$ again given by (4.3) and

$$\zeta_m = N^{(a''b'')}_{y''i''}(lmi''_0\ i_0-i'_0-m|i''\ i_0-i'_0)(2l)!\left[\binom{2l}{l+m}\binom{2i''_0}{n+m}\right]^{-1/2}.$$

Finally the general IFs are to be computed from the highest ones by repeatedly employing the recursion relation (cf, e.g. Hecht 1965)

$$(a'b'\ y'i'\ a''b''\ y''i''\|ab\ yi\ s)$$

$$= N_{yi}^{(ab)}(2\varkappa+1)^{1/2}\sum_{\tilde{i}}(-1)^{i_0+\varkappa+\tilde{i}+\frac{1}{2}}[(2\tilde{i}+1)/N_{y+1\tilde{i}}^{(ab)}]\begin{Bmatrix} i_0 & \varkappa & i \\ \frac{1}{2} & \tilde{i} & \varkappa-\frac{1}{2}\end{Bmatrix}$$

$$\times\Bigg(\sum_{\tilde{i}'}(-1)^{i'+i''+\tilde{i}+\frac{1}{2}}\langle a'b'\ y'i'\|B^{(1/2)}\|a'b'\ y'+1\ \tilde{i}'\rangle$$

$$\times(a'b'\ y'+1\ \tilde{i}'\ a''b''\ y''i''\|ab\ y+1\ \tilde{i}s)\begin{Bmatrix} i'' & i & i' \\ \frac{1}{2} & \tilde{i}' & \tilde{i}\end{Bmatrix}$$

$$+\sum_{\tilde{i}''}(-1)^{i'+\tilde{i}''+i+\frac{1}{2}}\langle a''b''\ y''i''\|B^{(1/2)}\|a''b''\ y''+1\ \tilde{i}''\rangle$$

$$\times(a'b'\ y'i'\ a''b''\ y''+1\ \tilde{i}''\|ab\ y+1\ \tilde{i}s)\begin{Bmatrix} i' & i & i'' \\ \frac{1}{2} & \tilde{i}'' & \tilde{i}\end{Bmatrix}\Bigg) \tag{4.6}$$

where $\varkappa=\frac{1}{2}(y_0-y)$; the explicit expression of general IFs in terms of the highest ones, particularly useful in checking the computed values, is

$$(a'b'\ y'i'\ a''b''\ y''i''\|ab\ yi\ s)$$

$$=(-1)^{i_0+\varkappa-i}N_{yi}^{(ab)}[(2i_0+1)(2\varkappa+1)]^{1/2}$$

$$\times\sum_{\substack{\tilde{y}'\tilde{i}'\\ \tilde{y}''\tilde{i}''}}\langle a'b'\ y'i'\|B^{(k')}\|a'b'\ \tilde{y}'\tilde{i}'\rangle\langle a''b''\ y''i''\|B^{(k'')}\|a''b''\ \tilde{y}''\tilde{i}''\rangle$$

$$\times\binom{2\varkappa}{2k'}\begin{Bmatrix} i_0 & \varkappa & i \\ \tilde{i}' & k' & i' \\ \tilde{i}'' & k'' & i''\end{Bmatrix}(a'b'\ \tilde{y}'\tilde{i}'\ a''b''\ \tilde{y}''\tilde{i}''\|ab\ y_0i_0\ s) \tag{4.7}$$

with $k'=\frac{1}{2}(\tilde{y}'-y')$. The formulae used in the algorithm follow from the completeness of the ξ basis, from the unraisability of the highest reduced states, and from the recursion relations (2.10) between the canonical states of different hypercharge.

Let us note in concluding that the s-classified CGCs with $(ab)=(00)$ (and, necessarily, $(a''b'')=(b'a')$, $(yii_z)=(000)$ and $s=2f_3(a'b')$) are given simply by

$$(a'b'\ y'i'i'_z\ a''b''\ y''i''i''_z|00\ 000\ s)$$

$$=\delta_{y',-y''}\delta_{i'i''}\delta_{i'_z,-i''_z}\frac{(-1)^{a'+b'+\frac{3}{2}y'+i'_z}}{[\dim(a'b')]^{1/2}}. \tag{4.8}$$

5. The s-classified SU(3) Clebsch–Gordan coefficients: the symmetry properties

Starting from the symmetry relations (3.2) imposed on the labelling operator, and from the phase convention (3.10) imposed on the s-classified highest reduced states, it is possible to show that the allowed s-classified SU(3) invariants of the products $D^{(a'b')}\times D^{(a''b'')}\times D^{(a'''b''')}$ and $\bar{D}^{(b'a')}\times\bar{D}^{(b''a'')}\times\bar{D}^{(b'''a''')}$, defined by equations analogous to

$$|a'b'\ a''b''\ a'''b'''\ s\rangle=\sum_{y'''i'''i'''_z}|(a'b'\ a''b'')b'''a'''\ -y'''i'''-i'''_z\ s\rangle|a'''b'''\ y'''i'''i'''_z\rangle\frac{(-1)^{b'''+a'''-\frac{3}{2}y'''-i'''_z}}{[\dim(b'''a''')]^{1/2}} \tag{5.1}$$

are linked by the symmetry relations

$$|a'b'\ a''b''\ a'''b'''\ s\rangle(-1)^{a'+b'+a''+b''+a'''+b'''}$$
$$=|a''b''\ a'b'\ a'''b'''\ -s\rangle=|a'b'\ a'''b'''\ a''b''\ -s\rangle$$
$$=\overline{|b'a'\ b''a''\ b'''a'''\ -s\rangle}. \qquad (5.2)$$

Accordingly, if the reduced state of the label s and of HW $(b'''a''')$ can be constructed by reducing the product $D^{(a'b')}\times D^{(a''b'')}$, then the reduced states of the opposite label $-s$ and of HWs $(b'''a''')$, $(b''a'')$ and $(a'''b''')$ can be constructed by reducing the products $D^{(a''b'')}\times D^{(a'b')}$, $D^{(a'b')}\times D^{(a'''b''')}$ and $D^{(b'a')}\times D^{(b''a'')}$, respectively.

Taking the scalar products of (5.2) with the relevant unreduced states, we find that the s-classified SU(3) CGCs satisfy the symmetry relations

$$(a'b'\ y'i'i'_z\ a''b''\ y''i''i''_z|b'''a'''\ -y'''i'''-i'''_z\ s)$$
$$=(a''b''\ y''i''i''_z\ a'b'\ y'i'i'_z|b'''a'''\ -y'''i'''-i'''_z\ -s)(-1)^{a'+b'+a''+b''+a'''+b'''}$$
$$=(a'b'\ y'i'i'_z\ a'''b'''\ y'''i'''i'''_z|b''a''\ -y''i''-i''_z\ -s)$$
$$\times(-1)^{a'+b'+\frac{3}{2}y'+i'_z}\left(\frac{\dim(a''b'')}{\dim(b'''a''')}\right)^{1/2}$$
$$=(b'a'\ -y'i'-i'_z\ b''a''\ -y''i''-i''_z|a'''b'''\ y'''i'''i'''_z\ -s)(-1)^{a'+b'+a''+b''+a'''+b'''}. \qquad (5.3)$$

The corresponding basic symmetry relations for the s-classified SU(3) IFs are then

$$(a'b'\ y'i'\ a''b''\ y''i''\|b'''a'''\ -y'''i'''\ s)$$
$$=(a''b''\ y''i''\ a'b'\ y'i'\|b'''a'''\ -y'''i'''\ -s)(-1)^{a'+b'+a''+b''+a'''+b'''+i'+i''-i'''}$$
$$=(a'b'\ y'i'\ a'''b'''\ y'''i'''\|b''a''\ -y''i''\ -s)(-1)^{a'+b'+\frac{3}{2}y'+i'}$$
$$\times\left(\frac{\dim(a''b'')(2i'''+1)}{\dim(b'''a''')(2i''+1)}\right)^{1/2}$$
$$=(b'a'\ -y'i'\ b''a''\ -y''i\|a'''b'''\ y'''i'''\ -s)(-1)^{a'+b'+a''+b''+a'''+b'''+i'+i''-i'''}. \qquad (5.4)$$

Thus the s-classified SU(3) CGCs exhibit the symmetry properties similar to those of their SU(2) counterparts. The coefficients linked by the basic symmetry relations of interchange, crossing and conjugation are those referring to opposite multiplicity labels, which is different from what happens in the SU(3) theory discussed in Derome (1967) and Butler (1975). The question of possible higher symmetries (cf Regge 1958) requires a special investigation.

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